A Method for Determining the Weak Statistical Stationarity of a Random Process

Willy Z. Sadeh* and Chester A. Koper Jr.†
Colorado State University, Fort Collins, Colo.

A method for determining the weak statistical stationarity of a random process is presented. The core of this testing procedure consists of generating an equivalent ensemble which approximates a true ensemble. Formation of an equivalent ensemble is accomplished through segmenting a sufficiently long time history of a random process into equal, finite, and statistically independent sample records. The weak statistical stationarity is ascertained based on the time invariance of the equivalent-ensemble averages. Comparison of these averages with their corresponding time averages over a single sample record leads to a heuristic estimate of the ergodicity of a random process. Specific variance tests are introduced for evaluating the statistical independence of the sample records, the time invariance of the equivalent-ensemble autocorrelations, and the ergodicity. Examination and substantiation of these procedures were conducted utilizing turbulent velocity signals.

Introduction

THE prime objective of an experiment is to supply data that can be efficiently utilized to develop and substantiate theoretical modeling of the physical phenomenon being investigated. Any observed data representing a physical phenomenon are generally classified as being either deterministic or nondeterministic. A physical process that is reasonably described by an explicit mathematical relationship so that its behavior can be predicted for a given set of conditions is deterministic. Physical processes that are not amenable to description by means of explicit mathematical relationships are termed nondeterministic or random. Examples of random processes are turbulent flow, aircraft acceleration, aircraft vibrations, wind, wind loading on structures, rainfall, runoff, waterwaves, and numerous other phenomena in engineering, physical sciences, natural sciences, communication, control, social sciences, and in many other fields. The salient trait of a random process is the lack of regularity in its behavior. For that matter, exact mathematical prediction of its outcome at a future instant of time is not possible. Description of a random process is feasible solely by means of its statistical properties. These statistical characteristics exhibit distinct regularity and, hence, they are employed to interpret the random process. A random process that can be described in terms of probability laws and statistical averages is commonly called a stochastic‡ process. In this paper both terms random and stochastic are used in the sense defined by the latter. Random data are usually expressed as a function of time. Any other variable can be used depending on the specific requirement. In practice, the statistical averages of a stochastic process amount simply to time averages. Random processes are categorized as being either statistically stationary or nonstationary. This division is based on the time invariance or time variation, respectively, of the statistical ensemble-averaged properties.

The issue of interest in analyzing random data is the interpretation of the time averages to insure that they supply a germane representation of the physical properties of the stochastic process being investigated. Adequate description of a process requires, in principle, an infinite number of its time histories (or physical realizations) which are supposed to be of infinite time length. They are the sample functions of the process. It is evident that acquisition of such sample functions is not feasible. Then it is conceivable to secure a limited number of realizations over finite time intervals. Such time histories are called sample records. All the possible sample records must be gathered simultaneously in independent and exactly similar physical situations using identical measuring instrumentation. The collection of all the sample records constitutes an ensemble. A hypothetical ensemble formed of N sample records of finite time length T_r of turbulent velocity $u_k(t)$, where $k=1\ldots N$, is portrayed in Fig. 1. An ensemble is denoted by a pair of braces, i.e., the ensemble of the sample records, $u_k(t)$, is designated by $\{u(t)\}$.

The properties of a random process are estimated by its ensemble averages (or ensemble moments) computed at selected times.[1] To start with, the ensemble mean value, i.e., the ensemble first-order moment, at a specific instant of time t_i, called starting time, is defined by

$$\langle\mu(t_i)\rangle=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}u_k(t_i) \tag{1}$$

where the subscript k designates each time history of the N sample records in the ensemble $\{u(t)\}$ and the bent brackets denote ensemble average. The ensemble autocorrelation, i.e., the ensemble second-order moment, at the same starting time t_i is

$$\langle R(t_i,t_i+\tau)\rangle=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}u_k(t_i)u_k(t_i+\tau) \tag{2}$$

in which τ is the time displacement. All other ensemble higher order moments and joint moments are computed in a similar manner. The computation of the foregoing two ensemble moments for a finite number of sample records N is illustrated in Fig. 1.

A stochastic process is weakly stationary in a statistical sense[1,2] whenever the ensemble mean value and autocorrelation are invariant with changing starting time t_i. If

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*Professor of Engineering, Fluid Mechanics and Wind Engineering Program, Department of Civil Engineering. Associate Fellow AIAA.

†Presently, Senior Engineer, Boeing Commercial Airplane Company, Seattle, Wash. Member AIAA.

‡The word "stochastic" is of Greek origin and initially had in English the meaning "conjecture." Apparently, the use of the term "stochastic" in relation to random processes was first advanced by the Swiss mathematician and physicist Jakob Bernoulli (1654-1705) in the 17th century.

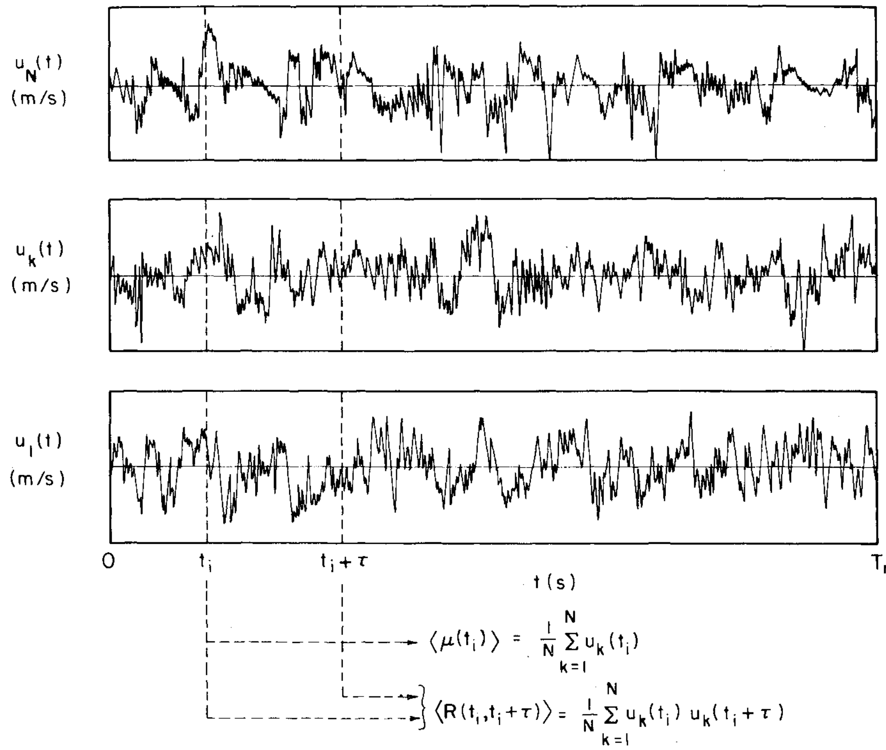

Fig. 1 Hypothetical ensemble of turbulent velocity sample records $\{u(t)\}$.

all higher-order ensemble moments exhibit the same starting-time invariance property, the random process is strongly stationary (or strictly stationary). A process is hence statistically nonstationary when the time invariance criterion of the ensemble moments is not fulfilled. Thus, the conditons for a weakly stationary process are

$$\langle \mu(t_i) \rangle = \langle \mu \rangle \tag{3}$$

and

$$\langle R(t_i, t_i+\tau) \rangle = \langle R(\tau) \rangle \tag{4}$$

A stochastic process is said to be covariance stationary if the starting-time invariance criterion for the autocorrelation expressed by Eq. (4) is satisfied without necessarily meeting the condition of a constant mean value.[3] The autocorrelation differs from the covariance function only for a nonzero mean value, otherwise they are equal. This is exactly the situation for turbulent velocity since its mean value is zero by definition. A covariance stationary process with zero mean value is therefore called *autocorrelation stationary*. For such a process the ensemble autocorrelation is solely a function of the time displacement τ.

A true ensemble cannot be obtained due to the insurmountable practical difficulties associated with the collection of the sample records. A single sample record $u_k(t)$ of a stochastic process is generally utilized for computing its statistical properties. This amounts to simply estimating the time-averaged characteristics of a selected realization. The temporal mean value and autocorrelation of any kth time history are

$$\mu(k) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u_k(t)\,dt \tag{5}$$

and

$$R(\tau,k) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u_k(t)\,u_k(t+\tau)\,dt \tag{6}$$

where T is the averaging time which is finite in practice. All higher-order time-averaged moments and joint moments are

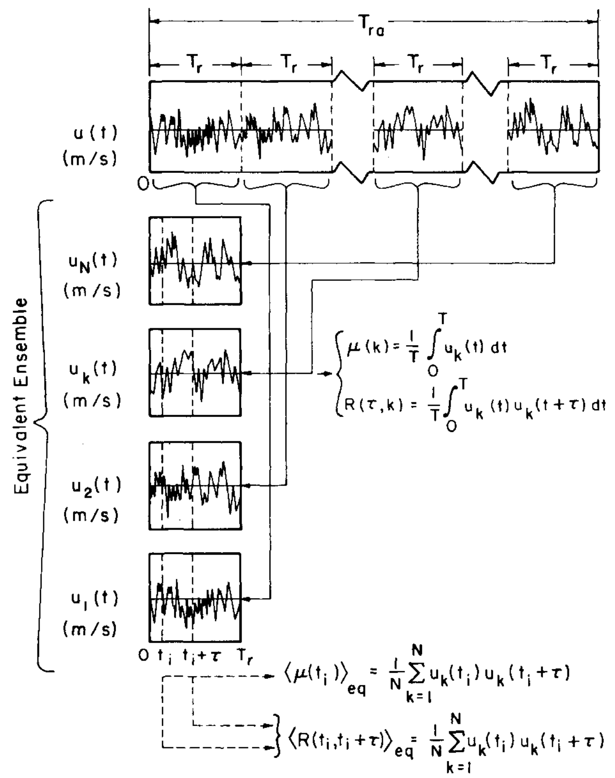

Fig. 2 Illustration of the generation of an equivalent ensemble $\{u(t)\}_{eq}$, from an available time history T_{ra}.

computed in a similar fashion. The intrinsic difference between ensemble averaging and single sample-record time averaging is that the former describes every conceivable state of the random process while the latter represents merely the properties of a particular realization of the process. Justification for the use of the temporal averages of a single sample record for determining the properties of a stochastic process is provided by the ergodic theorem.[3-5] A discussion of this theorem is beyond the scope of this paper. Ergodicity implies essentially that: 1) the time-averaged moments do not differ when computed over different sample records; and 2) these temporal moments can ultimately be identified with their corresponding ensemble averages. One can consequently select any realization for determining the properties of a stochastic process. The selection of a sample record is based, in practice, upon the reproducibility of the data in a controlled experiment.

It is important to point out that an ergodic process must be statistically stationary. At least, it must be weakly stationary. The converse is not necessarily true; a stationary process can be nonergodic. As a matter of fact, the ergodic theorem was first advanced with the introduction of harmonic analysis of stationary processes. Only for stationary processes are the autocorrelation function and the power spectral density function of a sample record related by Fourier transform pairs, i.e., the Wiener-Khintchine relations.[3]

In conclusion, for a weakly stationary ergodic stochastic process $\langle \mu \rangle = \mu(k) = \mu$ and $\langle R(\tau) \rangle = R(\tau, k) = R(\tau)$. Ascertainment of the statistical stationarity of a random process is hence of prime significance for assuring the reliability of the computed statistical averages under the ergodic assumption. Verification of the statistical stationarity can furthermore lead to a practical test regarding the assumed ergodicity of a process. A method for testing the statistical stationarity of a stochastic process is put forward herein. In addition, a heuristic test for the ergodic assumption is introduced. Turbulent velocity measured in a recent study of turbulent transport[6] is utilized for examining and substantiating the method advanced.

Equivalent Ensemble

Determination of the statistical stationarity of a random process is contingent upon the availability of an ensemble. Since a true ensemble cannot be acquired, it is proposed to approximate it by means of an *equivalent ensemble*. Such an equivalent ensemble can be formed by segmenting a sufficiently long time history into a finite number of sample records of equal time length. An equivalent ensemble, nevertheless, has to fulfill the underlying three conditions required for the creation of a true ensemble. These three criteria are: 1) all sample records are to be obtained under unchanged extraneous physical conditions; 2) all sample records must be longer than the largest time scale of interest; and 3) all sample records are statistically independent.

It is assumed that a long realization, $u(t)$, of a stochastic process is secured ensuring that all physical circumstances extraneous to the process remain unaltered throughout its acquisition. Grid turbulence at a selected velocity in a wind tunnel or vibrations in an aircraft at a fixed setup of the engines are examples of such physical situations. This single long observation represents the total available time history of the process and its length is denoted by T_{ra}. One can subsequently divide this available time history into N equal sample records, each of time length

$$T_r = T_{ra}/N \tag{7}$$

Every kth sample record is defined, in terms of the original time history $u(t)$, by

$$u_k(t) = u[t + (k-1)T_r] \tag{8}$$

in which $(k-1)T_r < t \leq kT_r$, $k = 1,2,\ldots,N$. All the N sample records are thus derived under similar physical conditions according to the first criterion for a true ensemble.

The time length of each sample record T_r is not known a priori. It can be estimated, to a first approximation, to equal the averaging time required to compute the time-averaged autocorrelation given by Eq. (6), i.e., $T_r \simeq T$. This approximation necessarily insures that the length of the sample records is greater than the largest time scale of interest. In the case of turbulence the latter is, for instance, the integral time scale which is determined in terms of the time-averaged autocorrelation. All the significant information regarding the random process is consequently comprehended within each sample record. This time length selection insures the fulfillment of the second condition for a true ensemble. Knowledge of the time length of each sample record leads to the creation of an equivalent ensemble. It is simply constructed by arranging the collection of all the sample records $u_k(t)$, each of length T_r, in the format characteristic to a true ensemble. Generation of such an equivalent ensemble $\{u(t)\}_{\text{eq}}$ is illustrated in Fig. 2.

Every sample record $u_k(t)$ must be statistically independent of all other sample records in the set $\{u(t)\}_{\text{eq}}$ according to the third criterion for a true ensemble. This condition entails satisfying the relationship

$$p_{k,m}(\xi,\eta) = p_k(\xi)p_m(\eta) \tag{9}$$

where $p_{k,m}(\xi,\eta)$ is the joint probability density function (JPDF) of any two sample records $u_k(t)$ and $u_m(t)$, and $p_k(\xi)$, $p_m(\eta)$ are the individual probability density functions (PDF) of the very same two sample records. Thus, the subscripts k and m take on different integer values from 1 to N, i.e., $k,m = 1,2,\ldots,N$, $k \neq m$. In the foregoing equation ξ and η denote the amplitudes of the sample records. Verification of the statistical independence of the sample records is essential for assuring the validity of the equivalent ensemble. This is accomplished by means of a suitable variance test.

Once an equivalent ensemble is created, one can compute its mean value and autocorrelation—i.e., $\langle \mu(t_i) \rangle_{\text{eq}}$ and $\langle R(t_i, t_i + \tau) \rangle_{\text{eq}}$, called hereinafter EEMV and EEAC for convenience—using Eqs. (1) and (2). Their computation is portrayed in Fig. 2. The test for weak stationarity is then similar to that for a true ensemble. When the equivalent-ensemble mean value and autocorrelation (EEMV and EEAC) do not vary as the starting time t_i changes, the stochastic process is approximated as being weakly stationary. Determination of the degree of starting-time invariance of these two moments is readily carried out by means of an adequate variance test. With the completion of the assessment of the statistical stationarity, the ergodic assumption can be examined by comparing the equivalent-ensemble averages with their corresponding time averages over a particular sample record. Calculation of the temporal mean value and autocorrelation is illustrated in Fig. 2. The comparison between the equivalent ensemble and the temporal averages is performed through an appropriate variance test.

Illustrative Example

Generation of an equivalent ensemble was undertaken utilizing turbulent velocity data collected during a recent investigation of turbulent transport in the lower atmosphere.[6] Measurement of the fluctuating velocity was carried out by means of hot-wire anemometers. The fluctuating voltage output of each hot-wire anemometer $e(t)$ was recorded on FM magnetic tape. All the results are presented hereinafter in terms of the fluctuating voltage $e(t)$, which is proportional to the turbulent velocity according to the hot-wire anemometer relationship. With the creation of an equivalent ensemble, the autocorrelation stationarity of the fluctuating voltage was estimated since its mean value is zero. The ergodicity of the process was subsequently examined.

Generation of an Equivalent Ensemble

A total available time history $T_{ra} = 3600$ s was used in generating an equivalent ensemble. During this time interval no perceivable changes in the extraneous physical circumstance were discerned. All the significant extrinsic conditions were continuously monitored throughout the

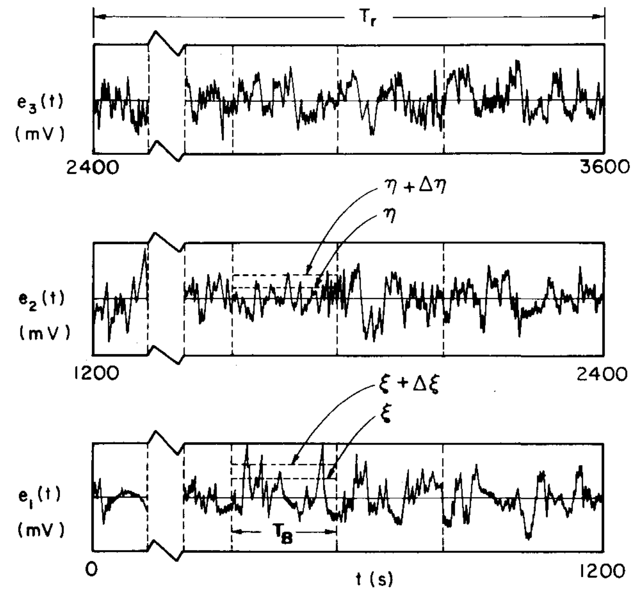

Fig. 3 Representation of the equivalent ensemble formed from an available time history T_{ra} of a fluctuating voltage $e(t)$ supplied by a hot-wire anemometer (not to scale).

measurement of the turbulent velocity. The selection of the total available time history length was hence in accordance with the first criterion for the establishment of an equivalent ensemble.

In line with the second requirement, the length of each sample record T_r was approximated to equal the averaging time T necessary to calculate the time-averaged autocorrelation. Based on the computation of the latter by means of Eq. (6), an averaging time $T = 1200$ s sufficed to encompass all the significant information concerning the turbulent velocity including its integral time scale. Three sample records, each of length $T_r = 1200$ s, were thus obtained by segmenting the total available time history in accordance with Eqs. (7) and (8). These three segments were:

$$e_1(t) = e(0 < t \leq 1200\,\mathrm{s}) \tag{10a}$$

$$e_2(t) = e(1200\,\mathrm{s} < t \leq 2400\,\mathrm{s}) \tag{10b}$$

and

$$e_3(t) = e(2400\,\mathrm{s} < t \leq 3600\,\mathrm{s}) \tag{10c}$$

An equivalent ensemble $\{e(t)\}_{\mathrm{eq}}$ consisting of these three sample records was henceforth created. This equivalent ensemble is portrayed in Fig. 3.

Examination of the remaining third essential criterion of statistical independence of the sample records involved computation of the JPDF's and the PDF's of the three sample records in the light of Eq. (9). Each JPDF and PDF was evaluated using the relationships

$$\hat{p}_{k,m}(\xi,\eta) = (1/\Delta\xi\Delta\eta)(T_{k,m}/T_B) \tag{11}$$

and

$$\hat{p}_k(\xi) = (1/\Delta\xi)(T_k/T_B) \tag{12}$$

where the hat indicates that these values represent estimators. In the foregoing first equation $T_{k,m}$ denotes the total amount of time that the amplitudes of both sample records $e_k(t)$ and $e_m(t)$ assume simultaneously values within the ranges $(\xi, \xi+\Delta\xi)$ and $(\eta, \eta+\Delta\eta)$, respectively. Similarly, T_k is the total amount of time that the amplitude of a single sample record $e_k(t)$ takes on values within the range $(\xi, \xi+\Delta\xi)$. These amplitude windows are shown in Fig. 3.

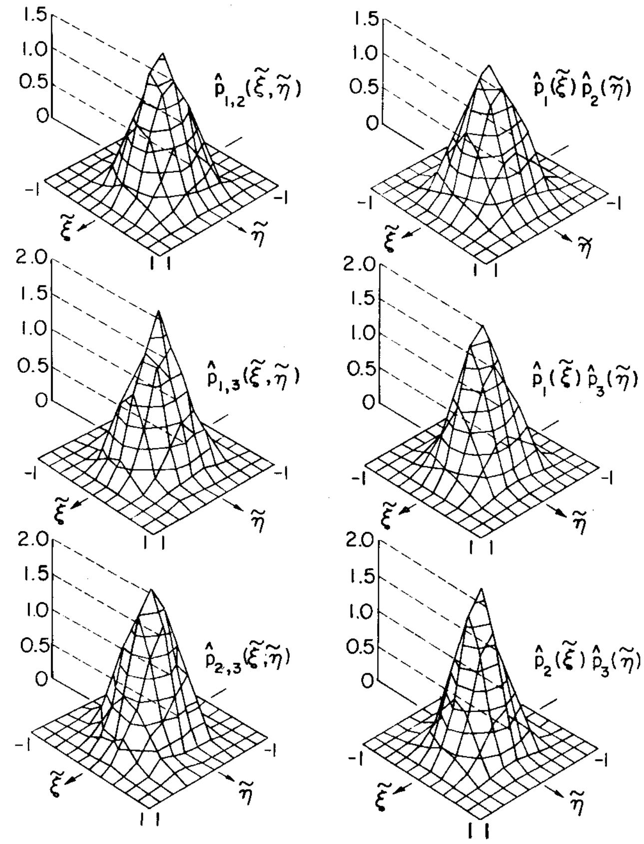

Fig. 4 Joint probability density functions $\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})$, and products of the single probability density functions $\hat{p}_k(\tilde{\xi})\hat{p}_m(\tilde{\eta})$ of the three sample records constituting the equivalent ensemble.

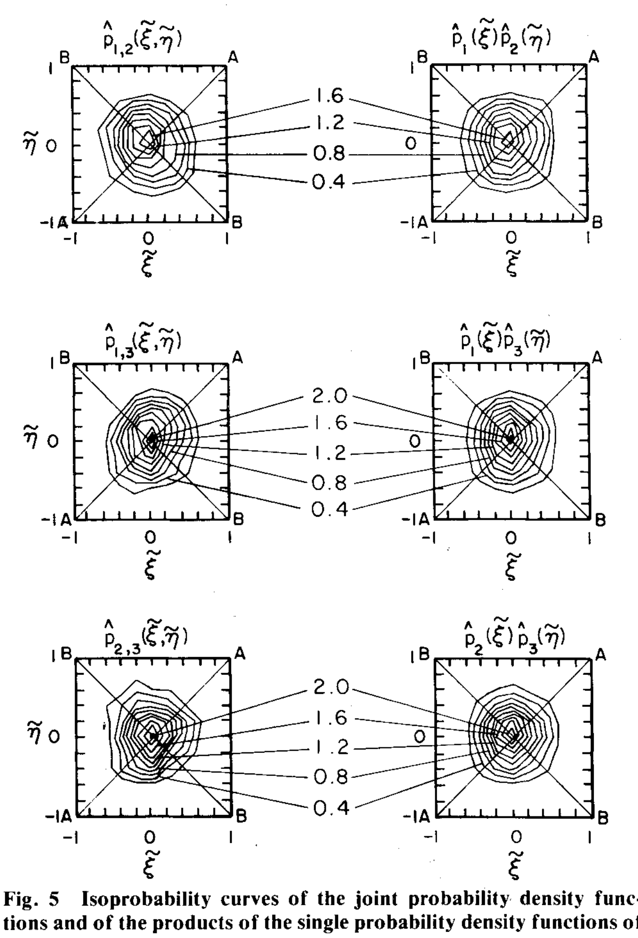

Fig. 5 Isoprobability curves of the joint probability density functions and of the products of the single probability density functions of the three sample records constituting the equivalent ensemble.

Both the JPDF's and the PDF's were evaluated using portions of the sample records extending over an observation time $T_B = 5$ s. The observation time T_B was determined based on the maximum time delay τ_{max} employed in calculating the time-averaged autocorrelation given by Eq. (6). This maximum time delay was 3 s, but a larger observation time interval of 5 s was selected. The observation time segments are illustrated in Fig. 3.

To carry out efficiently the computation of the JPDF's and the PDF's, the observation time segments were digitized at a sampling rate of 880 Hz yielding 4400 samples per segment. This sampling rate was 3.52 times greater than the highest frequency of interest of the signal, which was about 250 Hz. The latter was deduced from a discrete spectral analysis performed by means of a wave analyzer. In all cases more than 95% of the turbulent energy was contained up to a frequency of 250 Hz. In calculating the PDF's, the amplitudes ξ,η were made dimensionless employing the absolute value of the maximum instantaneous fluctuating voltage $|e_{max}|$. All the dimensionless amplitudes are indicated by a tilde. Throughout the computation of the PDF's, amplitude windows of 0.2, at most, were utilized.

The mean-square errors associated with the estimate of the JPDF's and the PDF's were evaluated. These errors amount only to the variance of the estimates since the bias error was negligible. The latter is proportional to the curvature of the JPDF's and PDF's which was, in all cases, sufficiently small to warrant discarding it. Calculation of the two normalized variance errors was carried out using the relationships

$$\epsilon_{p2}^2 \simeq \frac{c^2}{BT_B \Delta\tilde{\xi}\Delta\tilde{\eta}} \frac{1}{\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})} \tag{13}$$

and

$$\epsilon_{p1}^2 \simeq \frac{c^2}{2BT_B \Delta\tilde{\xi}} \frac{1}{\hat{p}_k(\tilde{\xi})} \tag{14}$$

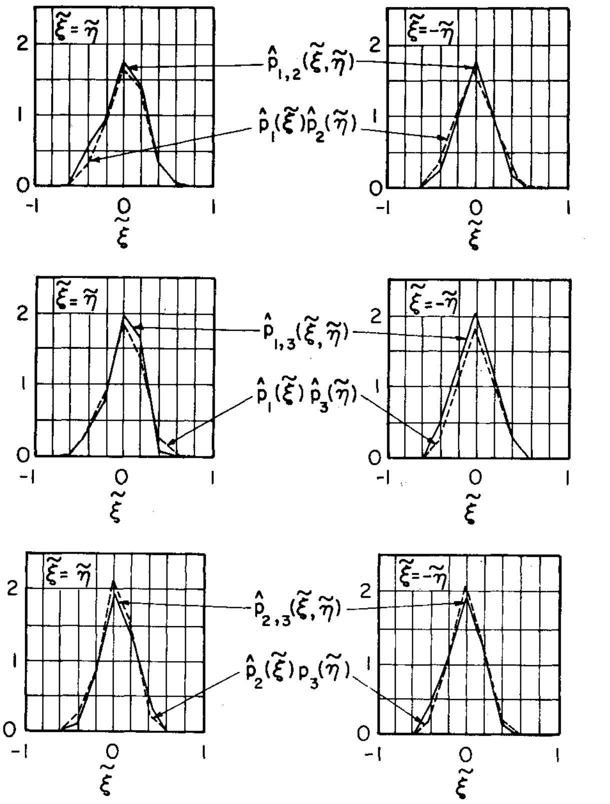

Fig. 6 Variations of the joint probability density functions and corresponding products of the single probability density functions along $\tilde{\xi} = \tilde{\eta}$ and $\tilde{\xi} = -\tilde{\eta}$.

for the JPDF and PDF, respectively.[1] In these two expressions the bandwidth of the signal is denoted by B and it is equal to the highest frequency of interest, i.e., $B = 250$ Hz. Since the number of digitized discrete points (4400) was larger than $2BT_B$ (2500), the constant c was assumed to equal unity.[1] The two variance errors depend upon the window widths and are restricted to nonzero values of JPDF's and PDF's. For a window width of 0.2, $\epsilon_{p2}^2 = 0.02/\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})$ and $\epsilon_{p1}^2 = 0.002/\hat{p}_k(\tilde{\xi})$. With augmenting values of $\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})$ and $\hat{p}_k(\tilde{\xi})$ from 0.1 to 2, ϵ_{p2}^2 decreased from 0.4 to 0.1 and ϵ_{p1}^2 from 0.04 to 0.001. The computed estimates of the JPDF's and the PDF's were therefore within a reasonable level of confidence.

The JPDF's and the product of the single PDF's define surfaces in the coordinate space $(\tilde{\xi},\tilde{\eta},\hat{p})$. Each set of three observation time segments, one from each sample record, yielded six surfaces. These six surfaces are shown in Fig. 4 and they are $\hat{p}_{1,2}$, $\hat{p}_{1,3}$, $\hat{p}_{2,3}$, $\hat{p}_1\hat{p}_2$, $\hat{p}_1\hat{p}_3$, and $\hat{p}_2\hat{p}_3$. A striking congruent variation between the JPDF's and their respective products of the PDF's is clearly discerned. This overall similarity is additionally corroborated by the isoprobability curves in the plane $\tilde{\xi}\tilde{\eta}$ portrayed in Fig. 5. Variations of both $\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})$ and $\hat{p}_k(\tilde{\xi})\hat{p}_m(\tilde{\eta})$ along the surfaces $\tilde{\xi} = \tilde{\eta}$ and $\tilde{\xi} = -\tilde{\eta}$ were examined to further ascertain the extent to which the statistical independence criterion put forth by Eq. (9) is met. Projections of these surfaces are indicated by the diagonals AA and BB in the $\tilde{\xi}\tilde{\eta}$ plane in Fig. 5. The variations of the JPDF's and corresponding product of the single PDF's in these planes are depicted in Fig. 6. A most remarkable quantitative agreement in their changes is perceived.

Quantitative assessment of the degree of statistical independence, according to Eq. (9), was estimated by computing the normalized variance between $\hat{p}_k(\tilde{\xi})\hat{p}_m(\tilde{\eta})$ and $\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})$. This normalized variance is defined by

$$(\tilde{\sigma}_p^2)_{k,m} = \frac{\int_{-1}^{1}\int_{-1}^{1} [\hat{p}_k(\tilde{\xi})p_m(\tilde{\eta}) - \hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})]^2 d\tilde{\xi}d\tilde{\eta}}{\int_{-1}^{1}\int_{-1}^{1} [\hat{p}_{k,m}(\tilde{\xi},\tilde{\eta})]^2 d\tilde{\xi}d\tilde{\eta}} \tag{15}$$

This variance expresses the relative volume that lies between the two surfaces formed by the JPDF and the product of the corresponding two PDF's. The variance test was applied to each set of observation time segments. A continuous decrease in the value of the variance was obtained with increasing observation time. For $T_B = 5$ s, the normalized variances were $(\tilde{\sigma}_p^2)_{1,2} = 0.0144$ and $(\tilde{\sigma}_p^2)_{1,3} = (\tilde{\sigma}_p^2)_{2,3} = 0.0169$. These results indicate that the three sample records are satisfactorily statistically independent. An acceptable equivalent ensemble, $\{e(t)\}_{eq}$, was thus created since all the three necessary requirements were reasonably met.

Autocorrelation Stationarity

Determination of the autocorrelation stationarity of the equivalent ensemble consisted of computing the equivalent-ensemble autocorrelation coefficients (EEACC) and examining their independence with changing starting time t_i. The EEACC is given by

$$\langle \rho(t_i, t_i+\tau) \rangle_{eq} = \frac{\langle R(t_i, t_i+\tau) \rangle_{eq}}{\langle R(t_i) \rangle_{eq}} \tag{16}$$

where the equivalent-ensemble autocorrelation $\langle R(t_i, t_i+\tau) \rangle_{eq}$ (EEAC) is calculated according to Eq. (2) with $N = 3$ (see Fig. 2), and $\langle R(t_i) \rangle_{eq}$ is the EEAC at zero time delay (at $\tau = 0$). Computation of the EEACC's was performed employing the same time segments, each of $T_B = 5$ s in length, used in the statistical independence test. Variation of the starting time t_i was confined within a range of 2 s. This range was singled out in the view of the maximum time delay,

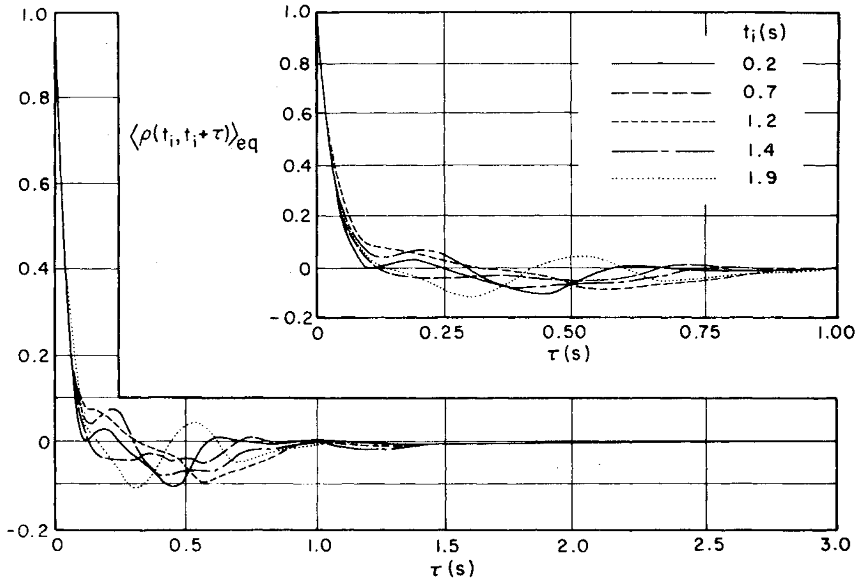

Fig. 7 Representative samples of the equivalent-ensemble autocorrelation coefficient at five selected starting times.

$\tau_{max} = 3$ s, prescribed by the time-averaged autocorrelation [Eq. (6)]. The same 3-s maximum time delay was utilized for computing the EEAC's. Inasmuch as a sampling rate of 880 Hz was utilized in the digitization of $e(t)$, 1760 EEACC's were obtained within the variation range of the starting time. A bandpass smoothing scheme was employed for editing purposes in computing the EEACC's. Data editing was necessary for detection and removal of any spurious and degraded signals that resulted from acquisition and recording problems.

Representative samples of the EEACC's at five selected starting times—viz., at $t_i = 0.2$, 0.7, 1.2, 1.4, and 1.9 s—are portrayed in Fig. 7. In addition, an expanded view of these five EEACC's during the first 1-s lag time is provided by the inset in Fig. 7. All the EEACC's reveal a similar variation regardless of their particular starting time. This notable congruence is indicative of their satisfactory independence of the starting time.

A variance test was further conducted for ascertaining to what degree the EEACC's were independent of the varying starting time. This test involved the computation of the variance of each EEACC within selected lag time increments about a unique autocorrelation of the whole equivalent ensemble. Such a particular autocorrelation is the average of all the EEACC's over all possible starting times. This equivalent-ensemble starting-time averaged autocorrelation coefficient (STAACC) is defined by

$$\langle\rho(\tau)\rangle_{eq} = \frac{1}{M}\sum_{i=1}^{M}\langle\rho(t_i,t_i+\tau)\rangle_{eq} \tag{17}$$

where M designates the total number of starting times. The STAACC is consequently a characteristic property of the entire equivalent ensemble.

In order to determine the independence of the EEACC's with respect to the varying starting time, an incremental variance within any desired lag time range, $\tau_1 \le \tau \le \tau_2$, was advanced. Computation of the normalized incremental variance was carried out using the relationship

$$\bar{\sigma}_\rho^2(t_i,\tau_1,\tau_2) = \frac{\int_{\tau_1}^{\tau_2}[\langle\rho(t_i,t_i+\tau)\rangle_{eq} - \langle\rho(\tau)\rangle_{eq}]^2 d\tau}{\int_0^{\tau_{max}}[\langle\rho(\tau)\rangle_{eq}]^2 d\tau} \tag{18}$$

in which $\tau_{max} = 3$ s. This incremental variance measures the contribution to the relative amount of area between the EEACC and the STAACC during any time delay interval $\Delta\tau = \tau_2 - \tau_1$. The normalized incremental variance was evaluated for constant lag time intervals $\Delta\tau = 0.1$, 1, 2, and 3 s. For the 0.1-s interval, the variance varied randomly within the range 0 to roughly 0.0196. When $\Delta\tau = 1$, 2, and 3 s, the variance changed systematically from 0.04 to 0.0784 with increasing starting time. These relatively higher values arose due to the accumulative effect embedded in the definition of the normalized incremental variance. These results lead distinctly to the inference that the EEACC's are reasonably dependent solely on the time displacement τ. It is conjectured that the consistency of the EEACC's starting-time invariance would be considerably enhanced with an increasing number of sample records. The equivalent ensemble consequently represented the realization of an autocorrelation stationary stochastic process since the EEACC's are reasonably time independent. In other words, one can deduce that the turbulent velocity was approximately weakly stationary. An equivalent ensemble can thus be utilized for ascertaining the autocorrelation stationarity and, therefore, the weak stationarity of a random process.

Ergodicity Test

Acquisition of a statistically stationary equivalent ensemble and knowledge of the EEACC's naturally suggest a test of the ergodic assumption. This test simply entails comparison of the EEACC with the temporal autocorrelation coefficient of a selected single sample record $R(\tau,k)$. The latter is given by Eq. (6) (see also Fig. 2). In view of the large number of available EEACC's it is desirable to use a single representative EEACC. This requirement is met by STAACC which is a unique feature of the entire equivalent ensemble. The temporal autocorrelation coefficient of any particular sample record is

$$\tilde{R}(\tau) = \overline{e(t)e(t+\tau)}/\overline{e^2(t)} \tag{19}$$

where the overbar designates time averaging. This autocorrelation coefficient was computed using Eq. (6) up to a maximum lag time of 3 s, as previously mentioned.

The STAACC, $\langle\rho(\tau)\rangle_{eq}$, and one of the time-average autocorrelation coefficients, $\tilde{R}(\tau)$, are depicted together in Fig. 8. An enlargement of both during the initial 1-s time delay is supplied by the inset in this figure. A prominent similarity in their variations is readily noticed.

A variance test was furthermore performed for evaluating to what extent the two autocorrelation coefficients are congruent. This variance test is utilized to approximate the

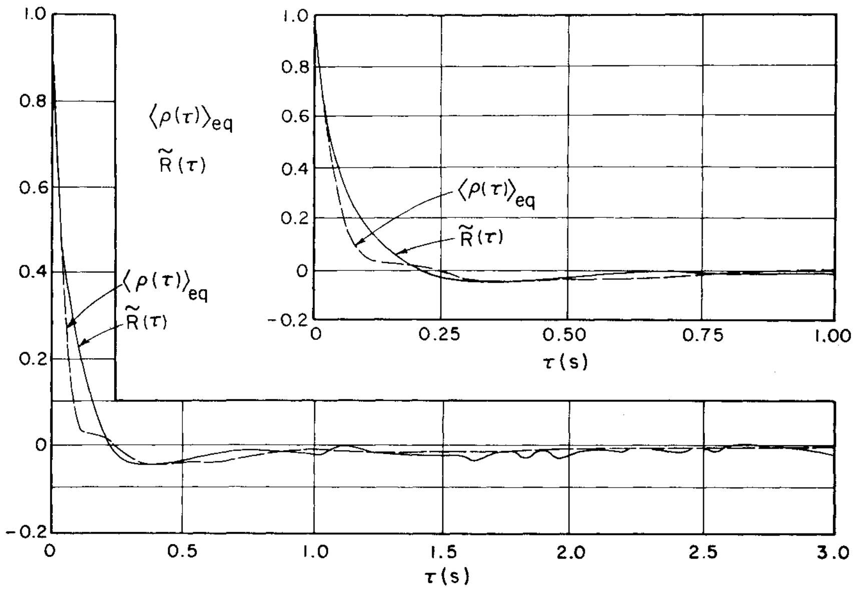

Fig. 8 Representative variations of the equivalent-ensemble starting-time averaged autocorrelation coefficient $\langle\rho(\tau)\rangle_{eq}$, and the single sample-record time-averaged coefficient $\tilde{R}(\tau)$.

validity of the ergodic assumption. A normalized incremental variance, formally similar to that of the stationarity test, was used for the examination of ergodicity. This variance is

$$\tilde{\sigma}_R^2(\tau_1,\tau_2) = \frac{\int_{\tau_1}^{\tau_2} [\langle\rho(\tau)\rangle_{eq} - \tilde{R}(\tau)]^2 d\tau}{\int_0^{\tau_{max}} \tilde{R}^2(\tau)dt} \tag{20}$$

It was estimated for the very same lag time intervals $\Delta\tau$ as the stationarity variance. For all $\Delta\tau = 0.1$-s intervals, the ergodicity variance $\tilde{\sigma}_R^2(\tau_1,\tau_2)$ changed irregularly in the range 0.0004 to 0.0121, at most. When $\Delta\tau = 1$, 2, and 3 s, the variance augmented systematically from 0.090 to about 0.137. These higher values are due, as for the stationarity variance, to the cumulative property of the incremental variance. It is quite remarkable, however, that the normalized incremental variance was smaller than about 0.0121 for all the 0.1-s lag time intervals. It is hypothesized that this satisfactory agreement between the two autocorrelation coefficients provides the support for accepting the assumption of ergodicity. One can thus infer that the turbulent velocity was approximately ergodic. The foregoing procedure demonstrates that the ergodicity of a stochastic process can be ascertained once the process is autocorrelation stationary.

Concluding Remarks

A method for testing the weak statistical stationarity of a random process is proposed. The procedure is based on the generation of an equivalent ensemble that represents a true ensemble. Creation of an equivalent ensemble is achieved by segmenting a sufficiently long time history of a random process into finite sample records of equal length. In forming an equivalent ensemble by the collection of these sample records, the basic three criteria necessary for producing a true ensemble are to be approximately fulfilled. These three conditions are: 1) all sample records are to be obtained under unchanged extraneous physical conditions; 2) all sample records ought to be longer than the largest time scale of interest for the process being investigated; and most importantly; 3) all the sample records must be mutually statistically independent.

The weak statistical stationarity of the process is determined based on the time invariance of the equivalent-ensemble mean value and the autocorrelation. Whenever only the latter condition is met and simultaneously the mean value is zero, the stochastic process is said to be autocorrelation stationary. Once the statistical stationarity of a process is established, one can inspect its ergodicity. This test involves comparison of the equivalent-ensemble averages with their corresponding time averages over any arbitrarily selected single sample record. Particular variance tests were introduced for estimating: 1) the statistical independence of the sample records forming the equivalent ensemble; 2) the time invariance of the equivalent-ensemble autocorrelation coefficients; and 3) the ergodicity of the autocorrelation coefficient. Incremental variances are put forward for testing the autocorrelation stationarity and the ergodicity. A unique autocorrelation coefficient characteristic of the entire equivalent ensemble is advanced in carrying out these incremental variance tests. This specific autocorrelation is obtained by time averaging all the equivalent-ensemble autocorrelations over their varying starting time.

The equivalent-ensemble method was examined using turbulent velocity data for substantiating its application. Both the autocorrelation stationarity and the ergodicity of the turbulent signal were reasonably ascertained. This successful verification of the equivalent-ensemble procedure indicates its reliability for determining the weak statistical stationarity and the ergodicity of a random process. It is apparent that the equivalent-ensemble method can be extended to higher order moments in order to estimate the strict statistical stationarity of a random process.

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